FLOW AND HEAT TRANSFER IN A FILM OF VISCOUS LIQUID ON A ROTATING DISK

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A numerical method employing a computer is used to investigate the hydrodynamics and heat transfer of a thin film of incompressible viscous liquid on a rotating disk for the case of laminar flow when it is possible to neglect surface tension, body forces, and friction between liquid film and gas medium.

Viscous liquid film flow over rotating surfaces needs to be studied in connection with the design of turbomachinery and various industrial equipment, as well as in other branches of technology. The published literature offers no theoretical solutions of this problem: Vachagin and Nikolaev [1] have attempted to obtain an approximate solution for average velocities, Espig and Hoyle [2] give the results of measurements of film thickness as a function of flow rate and disk speed for the wave flow regime. It was, therefore, considered desirable to make a theoretical study of the problem by solving the flow and energy equations numerically.

This paper is concerned with purely laminar flow without allowance for surface tension and associated wave effects. This influence of body forces and friction between liquid film and gas medium is neglected.

Assuming that the film is relatively thin, we can write the Navier-Stokes and energy equations in the form of equations of the boundary layer type [3]. In this case $\partial p/\partial y = 0$; in view of the rotation of the disk in an open space we also have $\partial p/\partial x = 0$. Therefore the flow and energy equations assume the form

$$\mathrm{Re}^2 u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial y} - \frac{v^2}{x} = \frac{\partial^2 u}{\partial y^2},$$
$$\mathrm{Re}^2 \left(u \frac{\partial v}{\partial x} + \frac{uv}{x} \right) + w \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial y^2},$$
$$\mathrm{Re}^2 \left(\frac{\partial u}{\partial x} + \frac{u}{x} \right) + \frac{\partial w}{\partial y} = 0,$$
$$\mathrm{Re}^2 \mathrm{Pr}\, u \frac{\partial \Theta}{\partial x} + \mathrm{Pr}\, w \frac{\partial \Theta}{\partial y} = \frac{\partial^2 \Theta}{\partial y^2}. \quad (1)$$

Here, $x = r/r_0$, $y = z/\delta_0$, $u = v_r/(r_0 \omega \mathrm{Re})$, $v = v_\varphi/(r_0\omega)$, $w = v_z s/\nu$, $\Theta = (T - T_s)/(T_d - T_s)$, $\mathrm{Re} = \delta_0^2 \omega/\nu$, $\mathrm{Pr} = \mu c_p/\lambda$.

Equations (1) are solved for the following boundary conditions (Fig. 1). At the initial radius $r = r_0$ the profiles of the velocity and temperature components are given, i. e., at $x = 1$

$$u = u^0(y),\ v = v^0(y),\ \Theta = \Theta^0(y). \quad (2)$$

From the condition of adhesion of the liquid to the rotating disk at $y = 0$

$$u = 0,\ v = x. \quad (3)$$

The absence of friction at the liquid-gas interface gives

$$\left.\frac{\partial u}{\partial y}\right|_s = \left.\frac{\partial v}{\partial y}\right|_s = 0. \quad (4)$$

At the surface of the disk the values of the normal component of velocity w are also given:

$$w|_{y=0} = w_0(x), \quad (5)$$

i. e., it is possible to specify any radial distribution of supply and removal of liquid through the surface of the rotating disk.

The temperature fields were calculated from the condition of constant temperature head, i. e.,

$$\Theta|_{y=0} = 1,\ \Theta|_s = 0. \quad (6)$$

Each of the second-order equations of system (1) is a nonlinear equation of parabolic type of the form

$$a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} + c, \quad (7)$$

where a, b, c depend on unknown functions.

We introduce the rectangular net

$$x = 1 + n\Delta x,$$
$$y = m \Delta y \ \ (n = 0,\ 1,\ \ldots;\ m = 0,\ 1,\ \ldots\ M(n))$$

and write (7) in difference form at the "half-integer" points

$$x = 1 + (n + \sigma)\Delta x,\ y = m\Delta y\ (0 \leqslant \sigma \leqslant 1).$$

We obtain the system of equations

$$a_m^{n+\sigma} \frac{f_m^{n+1} - f_m^n}{\Delta x} + b_m^{n+\sigma} \frac{1}{2\Delta y} [\sigma (f_{m+1}^{n+1} - f_{m-1}^{n+1}) + (1-\sigma)(f_{m+1}^n - f_{m-1}^n)] =$$
$$= \frac{1}{(\Delta y)^2} \{ [\sigma (f_{m+1}^{n+1} - f_m^{n+1}) + (1-\sigma)(f_{m+1}^n - f_m^n)] - [\sigma (f_m^{n+1} - f_{m-1}^{n+1}) + (1-\sigma)(f_m^n - f_{m-1}^n)] \} + c_m^{n+\sigma} \quad (8)$$

in the unknown functions f_m^n. In this case the values of the coefficients a, b, c at the "half-integer" points are determined by linear interpolation:

$$\varphi_m^{n+\sigma} = \sigma \varphi_m^{n+1} + (1-\sigma)\varphi_m^n. \quad (9)$$

We note that at $\sigma = 1/2$ we obtain the best order of approximation of the derivatives. In the linear case the same value ensures the stability of the computation process (at all $\sigma \geq 1/2$). Therefore the calculations were made for $\sigma = 1/2$.

Table 1

Change in Flow Parameters Along Radius with Variation in Flow Rate (for Re = 10, Pr = 2)

r/r_0	$\bar{\tau}_r$	$-\bar{\tau}_\varphi$	$\left(\frac{v_r}{v_{r0}}\right)_s$	$\left(\frac{v_\varphi}{r\omega}\right)_s$	$-(Nu)_d$
			$\varphi=0.5$		
1	20	0	1.00	1.00	20
1.1	6.90	1.06	1.31	0.826	4.37
1.2	7.02	1.61	1.48	0.695	3.56
1.3	7.28	2.06	1.59	0.595	3.20
1.4	7.52	2.45	1.63	0.526	2.97
1.5	7.76	2.80	1.64	0.481	2.81
1.6	7.96	3.10	1.62	0.457	2.68
1.7	8.14	3.35	1.59	0.450	2.58
1.8	8.32	3.56	1.56	0.453	2.50
1.9	8.50	3.73	1.53	0.466	2.45
2.0	8.74	3.86	1.50	0.488	2.43
			$\varphi = 1.5$		
1	20	0	1.00	1.00	20
1.1	5.14	1.42	1.04	0.826	5.43
1.2	3.72	2.05	1.07	0.694	4.28
1.3	4.10	2.55	1.09	0.591	3.77
1.4	3.97	2.97	1.10	0.510	3.45
1.5	3.90	3.35	1.11	0.445	3.23
1.6	3.87	3.69	1.12	0.391	3.07
1.7	3.85	4.00	1.12	0.348	2.94
1.8	3.86	4.29	1.11	0.315	2.84
1.9	3.88	4.57	1.10	0.288	2.75
2.0	3.90	4.84	1.08	0.268	2.68

Table 2

Effect of Reynolds Number on Radial Flow Development (at φ = 2.5, Pr = 2)

r/r_0	$\bar{\tau}_r$	$-\bar{\tau}_\varphi$	$\left(\frac{v_r}{v_{r0}}\right)_s$	$\left(\frac{v_\varphi}{r\omega}\right)_s$	$-(Nu)_d$	$\bar{\tau}_r$	$-\bar{\tau}_\varphi$	$\left(\frac{v_r}{v_{r0}}\right)_s$	$\left(\frac{v_\varphi}{r\omega}\right)_s$	$-(Nu)_d$
	Re = 10					Re = 50				
1	20.0	0	1.00	1.00	20.00	20.0	0	1.00	1.00	20.0
1.15	5.38	2.25	1.02	0.755	5.93	10.3	5.03	1.02	0.753	10.8
1.30	4.20	3.12	1.03	0.592	4.52	8.80	7.10	1.03	0.592	9.17
1.45	3.76	3.79	1.04	0.475	3.92	8.12	8.66	1.04	0.466	8.27
1.60	3.54	4.35	1.04	0.391	3.57	7.74	9.96	1.05	0.390	7.68
1.75	3.41	4.85	1.05	0.327	3.33	7.70	11.1	1.05	0.327	7.25
1.90	3.34	5.31	1.04	0.279	3.16	7.34	12.1	1.06	0.277	6.91
2.05	3.29	5.73	1.03	0.243	3.02	7.26	13.1	1.06	0.238	6.64
2.20	3.27	6.13	1.01	0.217	2.91	7.20	14.0	1.06	0.206	6.42
2.35	3.26	6.50	0.982	0.201	2.81	7.20	14.8	1.06	0.181	6.23
2.50	3.26	6.85	0.948	0.192	2.73	7.20	15.6	1.06	0.160	6.07
2.65	3.26	7.18	0.918	0.188	2.65	7.24	16.4	1.06	0.143	5.92
2.80	3.27	7.48	0.880	0.191	2.58	7.30	17.2	1.06	0.128	5.80
2.95	3.27	7.75	0.845	0.198	2.51	7.36	17.9	1.06	0.116	5.68
3.10	3.28	7.99	0.811	0.208	2.45	7.44	18.6	1.06	0.107	5.58
3.25	3.28	8.21	0.778	0.222	2.39	7.52	19.4	1.05	0.099	5.49
3.4	3.29	8.40	0.745	0.238	2.33	7.62	20.1	1.04	0.094	5.41
3.55	3.31	8.57	0.715	0.257	2.29	7.74	20.8	1.03	0.092	5.33
3.70	3.33	8.72	0.690	0.275	2.25	7.84	21.4	1.02	0.090	5.26
3.85	3.36	8.87	0.665	0.298	2.21	7.96	22.1	1.00	0.091	5.20
4.00	3.41	8.99	0.644	0.321	2.20	8.10	22.8	0.981	0.095	5.14

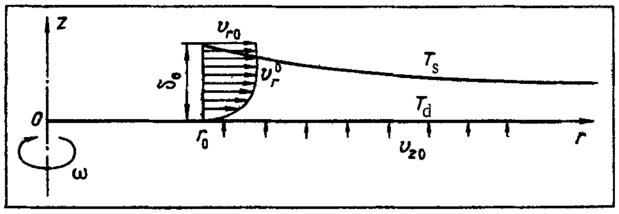

Fig. 1. Diagram of the problem.

In order to satisfy the continuity equation and the boundary condition for w, it is convenient to introduce the stream function

$$\psi = \int_0^y uxdy. \quad (10)$$

Then from the continuity equation the values of $w_m^{n+\sigma}$ are computed from

$$w_m^{n+\sigma} = w_0^{n+\sigma} + \frac{\mathrm{Re}^2}{\Delta x\,[1+(n+\sigma)\,\Delta x]}\,(\psi_m^n - \psi_m^{n+1}). \quad (11)$$

The thickness of the viscous film in the (n + 1)-th section is determined after obtaining the value of the total flow rate ψ_*^{n+1}, which depends on the flow rate through the initial section $\psi_*^0 = \int_0^1 u^0 dy$ and the rate of supply through the surface of the disk $w_0(x)$:

$$\psi_*^{n+1} = \psi_*^n + (w_0^{n+\sigma} - w_M^{n+\sigma})\,\frac{\Delta x\,[1+(n+\sigma)\,\Delta x]}{\mathrm{Re}^2}. \quad (12)$$

Having the values of u_m^n in the given section in some approximation, from (10) we compute the values of ψ_m^n up to some m = M(n) for which

$$\psi_{M-1}^{n+1} < \psi_*^{n+1} \leqslant \psi_M^{n+1}. \quad (13)$$

In each section the second-order difference Eq. (8) takes the form

$$\alpha_m^n f_{m-1}^n + \beta_m^n f_m^n + \gamma_m^n f_{m+1}^n = \delta_m^n. \quad (14)$$

This equation is solved by the pivotal condensation method: first the coefficients

$$A_{m-1}^n = -\alpha_{m-1}^n : (\beta_{m-1}^n + \gamma_{m-1}^n A_m^n),$$

$$B_{m-1}^n = (\delta_{m-1}^n - \gamma_{m-1}^n B_m^n) : (\beta_{m-1}^n + \gamma_{m-1}^n A_m^n), \quad (15)$$

are computed, and then the values of the function

$$f_{m+1}^n = A_{m+1}^n f_m^n + B_{m+1}^n. \quad (16)$$

For boundary conditions (4), with account for (13), using quadratic interpolation we obtain the following equations for the initial coefficients:

$$A_M^n = \left(\beta_{M-1}^n + 2\,\frac{q+1}{q+1/2}\,\alpha_{M-1}^n\right):$$

$$:\left(\frac{q+3/2}{q+1/2}\,\alpha_{M-1} - \gamma_{M-1}^n\right),$$

$$B_M^n = \delta_{M-1}^n : \left(\gamma_{M-1}^n - \frac{q+3/2}{q+1/2}\,\alpha_{M-1}\right),$$

$$q = (\psi_M^n - \psi_*^n) : (\psi_{M-1}^n - \psi_M^n). \quad (17)$$

Boundary conditions (6) for the temperature give

$$A_M^n = q : (1-q), \quad B_M^n = 0. \quad (18)$$

The quantities α_m^n, β_m^n, γ_m^n, δ_m^n depend not only on the values of the unknown functions in the previous section, but also on the values in the given section; therefore it is necessary to use iteration.

In order to work out and test the algorithm and the program, an analytic similar solution was found for small Re numbers. This solution is obtained from the condition of constant film thickness for a corresponding uniform supply of liquid through the disk. It has the form

$$u = u^0(y)\,x, \quad v = v^0(y)\,x, \quad (19)$$

where

$$u^0 = \left(y - \frac{1}{2}\,y^2\right) + \mathrm{Re}^2 y \times$$

$$\times\left(-\frac{1}{360}\,y^5 + \frac{1}{60}\,y^4 - \frac{1}{9}\,y^2 - \frac{1}{3}\,y + \frac{14}{15}\right) + \ldots,$$

$$v^0 = 1 + \frac{\mathrm{Re}^2}{12}\,y\,(-y^3 + 4y^2 - 8) + \ldots. \quad (20)$$

In this case medium is supplied at a rate

$$w_0 = \frac{2}{3}\,\mathrm{Re}^2 - \frac{208}{315}\,\mathrm{Re}^4 + \ldots. \quad (21)$$

Calculations based on this program with velocity distributions in the initial section taken in accordance with (20) and with uniform infiltration, in accordance with (21), give conservation of film thickness, nondependence of w on x, and a radial velocity variation proportional to x in complete conformity with (19).

We now present some results. They were obtained for $\Delta x = 0.005$, $\Delta y = 0.025$, and the iteration accuracy $\varepsilon = 10^{-5}$. The initial velocity and temperature profiles were assumed almost constant:

$$u^0 \approx u_0, \quad v^0 \equiv 1, \quad \Theta^0 \approx 0;$$

liquid was not supplied through the surface of the disk ($w_0 \equiv 0$).

The pattern of radial flow development (Fig. 2) shows that at a smaller flow parameter $\varphi = v_{r0}/(r_0\omega)$ the film thickness decreases more sharply owing to the more rapid increase in radial velocities. Variation of φ has a lesser effect on the behavior of the circular velocity and temperature profiles. With increase in the dimensionless radial velocity gradient v_r/v_{r0} at the disk, as φ decreases the dimensionless friction $\bar{\tau}_r$ increases correspondingly (Table 1). The dimensionless circular component of the friction stress on the disk $\bar{\tau}_\varphi$ varies less sharply with change in φ, like the dimensionless heat flux Nu. The radial velocity at the surface of the disk $(v_r/v_{r0})_s$, like the dimensionless circular velocity defect $(r\omega - v_\varphi)_s/r\omega$, at first increases along the radius and then begins to fall.

The effect of the Reynolds number Re at fixed flow rate (Fig. 3 and Table 2) is expressed in an acceleration of the decrease in film thickness with increase in Re as the radial velocity increases. Apart from the increase of $(v_r/v_{r0})_s$ with increase in Re, the profile of

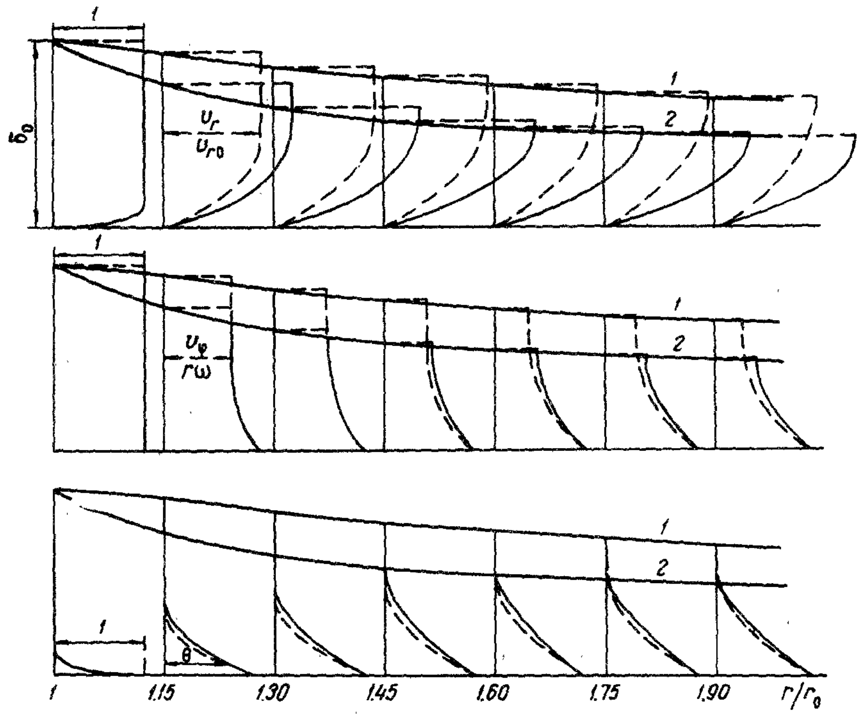

Fig. 2. Effect of flow rate on motion of liquid film at Re = 10 (Pr = 2). The solid lines and 1 correspond to $\varphi = 0.5$, the dashed lines and 2 to $\varphi = 1.5$.

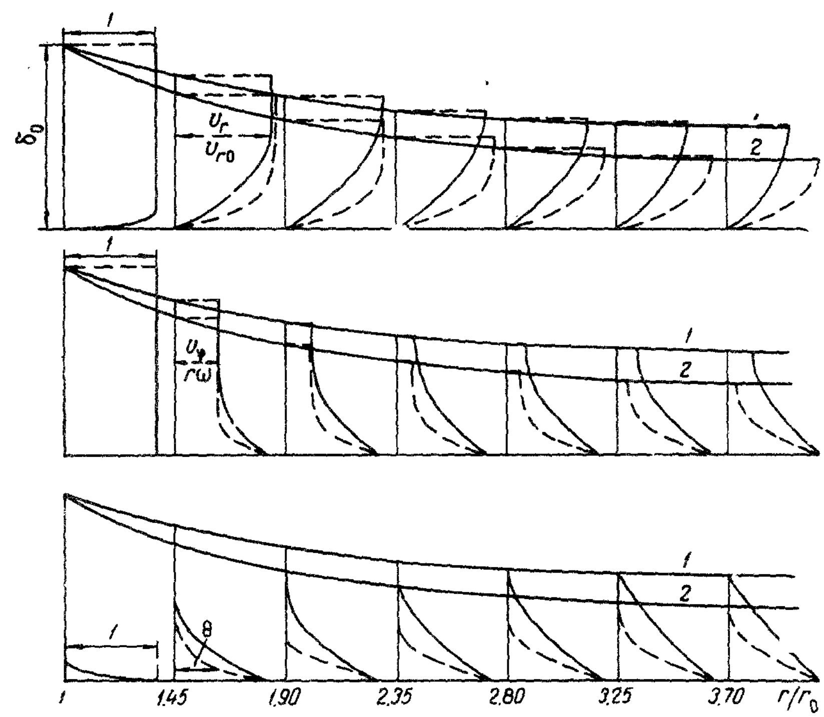

Fig. 3. Effect of Re number on motion of liquid film at $\varphi = 2.5$ (Pr = 2). The solid lines and 1 correspond to Re = 10, the dashed lines and 2 to Re = 50.

this velocity becomes fuller, the gradient at the disk increases. Correspondingly, as Re grows, so do the friction stresses. The same applies to the circular velocity defect $(r\omega - v_\varphi)_s : r\omega$.

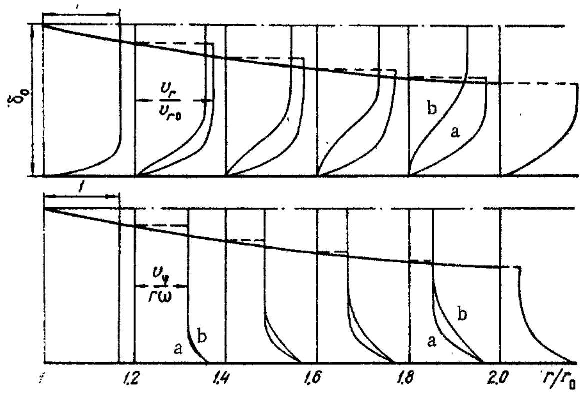

Fig. 4. Comparison of flow in liquid film on rotating disk (a) with flow in half-width of gap between two rotating disks (b) for the same $\varphi = 5$ and Re = 12.5.

We note that whereas when φ increases the coefficient of heat transfer from the disk increases only slightly, with increase in Re it grows substantially, almost as $Re^{1/2}$.

We will compare the flow in a viscous liquid film on a rotating disk with the flow between two identically rotating disks at identical values of $\varphi = v_{r0}/(r_0\omega) = 5$ and $Re = \delta_0^2 \omega/\nu = 12.5$ and the same inlet velocity profiles. In this case for flow between two disks δ_0 is the half-width of the gap. The velocity fields are also shown (Fig. 4) for the half-width of the gap. Calculation of the flow between two disks, which was based on another program [4], revealed the development of the radial velocity component into a separation profile (separation occurred at $x > 1.8$), whereas the flow of the liquid film was separationless, and the profiles of radial velocity and circular velocity defect were full. Accordingly, the friction stresses at the disk for a liquid film flow considerably exceed the values for the case of two disks.

NOTATION

v_r, v_φ, and v_z are the radial, circular, and normal velocity components, respectively; p is the pressure; T is the temperature; ρ is the density; ν is the kinematic viscosity; $\mu = \rho\nu$; c_p is specific heat; q is specific heat flux; ω is angular velocity; r_0 is the initial radius; δ_0 is the film thickness at initial radius; $\bar{\tau}_r = \tau_r s/\mu v_{r_0})$; $\bar{\tau}_\varphi = \tau_\varphi s/(r_0\omega\mu)$; $Nu = qs/\lambda\,(T_d - T_s)$. Subscripts: d—on disk, s—on film surface.

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